

# SEMI-PARAMETRIC VOLATILITY SURFACE ESTIMATION

by

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## **Abstract**

In this paper, we have investigated the performance of a semi-parametric model of volatility surfaces by applying it to the prices of exchange-traded options on crude oil futures in a highly volatile environment. We have also reviewed the performance of a parametric and a non-parametric model in such market conditions. To confirm our findings, we have provided some performance metrics as well as further analysis on the dynamics of the underlying futures. Based on our work, we have shown that the above models do not perform well in highly volatile conditions.

The overall contribution of this paper is to determine whether local volatility models in general are applicable to options on crude oil futures in a highly volatile market or we should move towards stochastic volatility approaches.

**Keywords:** Semi-parametric; Parametric; Non-parametric; Volatility surface; Local volatility; Stochastic volatility

## **Dedication**

To my parents who are my dearest...

## **Acknowledgements**

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# 1: Introduction

In this paper, we deal with three models of volatility surfaces. A semi-parametric model that was introduced in Borovkova and Permana (2009), a parametric model (Dumas et al., 1998) and a non-parametric model (Bourke, 1998). We have tested the performance of these models by applying them to options on crude oil futures during highly volatile market conditions.

Our focus in this project is mainly on the performance of the semi-parametric approach. The other two methods were already investigated in Borovkova and Permana (2009), but for comparison purposes and in order to test their feasibility in highly volatile market conditions, we have presented them here.

The crude oil futures market started a heavy downward trend towards the end of June 2014 and ever since and up to the date we have prepared this document, it has not rebounded to the previous levels. This downward trend was accompanied by high volatilities as well as more uncertainties.

As we will discuss in the literature review section, modeling implied volatilities in commodity markets, especially crude oil is less studied than other markets such as equity and foreign exchange. The semi-parametric model that we have tested here is one of the very few attempts to address this shortage.

In the semi-parametric approach, a quadratic function is fitted to implied volatilities at each maturity. We call each of these fitted quadratic functions a volatility slice. The coefficients of these quadratic functions are calculated through either the ordinary least-squares method or a weighted least-squares approach. After this stage, the volatility slices are connected to each other through bi-cubic spline interpolation to build the whole volatility surface. The advantage of a semi-parametric approach is that we can have different shapes of volatility slices at different maturities, such as a smile, a smirk or a skew and it provides a relatively good fit on at-the-money implied volatilities.

Based on our analysis we have shown that when volatilities reach record high levels in the market this method fails to provide a good fit to implied volatilities. This fact becomes more apparent when the number of liquid options increases and when we move away from at-the-money strike/moneyness. In addition, in this model it is implicitly assumed that market only behaves in

certain ways and a quadratic function is able to capture and depict those shapes. This is rather simplistic and restrictive, as we cannot force the market to behave in specific ways.

In order to verify our observations we have estimated some metrics and have compared them with the ones in Borovkova and Permana (2009). Our comparison verifies our findings and the fact that the performance of these models exacerbates when we are dealing with high volatility.

We have also conducted some work to see whether local volatility models in general are useful in modeling implied volatilities in the crude oil market. We have noted volatility clustering in the futures daily returns as well as high peak and fat tails in the distribution. These features along with the fact that daily returns on futures do not follow normal distribution suggest that the volatility itself should be considered as a stochastic process.

Traders usually use volatility surfaces as a tool to price options that do not exist on exchanges or to price OTC options with more complex structures. So in the modeling of the volatility surfaces the choice of model becomes an essential part that we suggest should be decided upon by considering the dynamics of the underlying asset. In the case of options on crude oil futures we believe stochastic volatility models would lead to better results.

This paper is structured as follows: For the remainder of this section we provide background knowledge and review some of the literature in volatility modeling. In section (2), we introduce the models that we have applied in our analysis. Section (3) covers our research methodology, data analysis and results. In section (4) we give our conclusion.

## **1.1 Background**

Volatility is a parameter that cannot be directly observed the same way as interest rates and stock prices. But the best thing we can do is to measure it using statistical methods. But such measure would be backward looking while we really want to take a forward-looking approach. In pricing options, we care about the actual volatility for the remaining life of the option. To deal with this situation, the best thing we can do is to use implied volatilities. Implied volatility is the market's best estimate of what volatility will do in the future. In this sense, the implied volatility of a 6-month option may contain information about how the actual volatility will behave during the next six months (Wilmott, 2007).

In a theoretical sense, implied volatility is the volatility of an underlying asset, which when substituted into an option pricing model like the Black-Scholes, gives a theoretical price equal to the market price of the underlying. The Black-Scholes (1973) formula for an option takes the time to maturity, interest rate, strike price, spot price of underlying together with volatility as input and generates the price as an output.

It has been extensively argued that assumptions underlying the Black-Scholes model, such as constant volatility and lognormal distribution of the underlying asset are not realistic. The Black-Scholes (1973) model, for example, assumes the asset price follows geometric Brownian motion with constant volatility. Consequently, all options on the same asset should provide the same implied volatility. In practice, however, Black-Scholes implied volatilities tend to differ across exercise prices and times to expiration (Dumas et al., 1998).

Traders still use the Black-Scholes model but not in the way it was originally introduced. They allow the volatility parameter to depend on the strike price and time to maturity. In this case, the most straightforward approach for modeling the implied volatility would be to use local volatility models that define volatility as a deterministic function of time and underlying's price (Wilmott, 2007) and (Hull, 2012).

Although local volatility models have some attractive features, but without any doubt we still cannot forecast the future actual volatility based on these models accurately. To further address these issues stochastic volatility models have been developed. Stochastic volatility is the notion that volatility varies in a random fashion (Gatheral, 2006). So in addition to modeling the underlying as random, in stochastic volatility models, the volatility parameter itself is also modeled as a random process. To link stochastic volatilities to local volatilities, Gatheral (2006) shows that local variance is a conditional expectation of the instantaneous variance.

A plot of the implied volatility of an option with a certain life as a function of its strike price is known as a volatility smile. And when the implied volatility at a specific strike price is plotted as a function of time to maturity we have volatility term structure. Volatility surfaces combine volatility smiles with the volatility term structures to tabulate the volatilities appropriate for pricing an option with any strike price and any maturity (Hull, 2012). Volatility surfaces are important in the sense that option traders use them as a tool in order to price options.

In this paper we deal with three types of modeling volatility surfaces: parametric, non-parametric and semi-parametric. In a parametric model, we build a volatility surface by just estimating a finite set of parameters. Using a non-parametric approach, the parameters would be in infinite-dimensional parameter spaces and finally a semi-parametric method is comprised of both finite-dimensional and infinite-dimensional unknown parameters of interest.

## **1.2 Literature review**

Modeling the volatility surfaces is subject of extensive research in the financial markets and more specifically the equity markets. But implied volatility is much less studied in the commodities

market due to limited liquidity and complicated structure of commodity options (Borovkova and Permana, 2009).

Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) hypothesized that asset return volatility is a deterministic function of asset price and time, and developed a deterministic volatility function (DVF) option valuation model that has the potential of fitting the observed cross section of option prices exactly. To verify this, Dumas et al. (1998) examined the predictive and hedging performance of the DVF option valuation model by using S&P 500 options from June 1988 through December 1993 and demonstrated that the DVF model is no better than an ad hoc procedure that merely smooths Black–Scholes (1973) implied volatilities across exercise prices and times to expiration.

In their article, Gatheral and Jacquier (2014) show how to calibrate the widely used SVI parameterization of the implied volatility smile in such a way as to guarantee the absence of static arbitrage. In particular, they exhibit a large class of arbitrage-free SVI volatility surfaces with a simple closed-form representation and demonstrate the high quality of typical SVI fits with a numerical example using SPX options data.

Murphy and Ronn (2014) calibrated the volatility skew by the Merton (1976) jump-diffusion option-pricing model using market prices for crude-oil futures options and the prices of their underlying futures contracts. They demonstrated that the jump-diffusion parameters bear a close relationship to concurrent economic, financial and geopolitical events.

Bouden (2007) investigated implied volatility surface patterns for call options on crude oil futures. They tested the capacity of quadratic Implied Volatility Functions (IVFs) to provide a good forecast of option prices and found that the profile of crude oil implied volatility is too complex to be fully explained by IVFs.

It is generally accepted that the implied volatilities from liquidly traded options, specifically at-the-money options, provide the best forecast of future volatility. In their paper Wang et al. (2009) showed that the degree of the bias of the implied volatility estimation is directly related to the option's moneyness and maturity and such bias can be minimized by choosing option with the highest Vega to estimate volatility. They used the S&P 500 index option to test the informational content of the implied volatility from options with highest Vega in comparison to that of at-the-money implied volatility and demonstrated that the implied volatility from options with highest Vega outperforms at-the-money implied volatility, especially for longer forecasting horizons.

## 2: Models

### 2.1 Parametric Model

The parametric model is a quadratic function of either option's strike ( $E$ ) or option's moneyness ( $MN$ ), and a linear function of time to maturity ( $\tau$ ). In addition, it captures the combined effect of strike/moneyness and time to maturity.

#### Moneyness:

Moneyness is a unit-less quantity defined as follows:

$$(2-1) \quad MN = \frac{\log\left(\frac{F(t, T)}{E}\right)}{\sigma\sqrt{\tau}}$$

In which the symbols used are:

$$\tau = T - t$$

$F(t, T)$  : Futures price (Here, Crude Oil Futures)

$E$  : Option's strike

$\sigma$  : Option's implied volatility

$\tau$  : Time to maturity of the option

$T$  : Maturity of the option

$t$  : Current time

For a call option, positive moneyness corresponds to In-The-Money options (ITM), negative moneyness corresponds to Out-of-The-Money options (OTM) and zero moneyness corresponds to At-The-Money options (ATM). And for a put option positive moneyness corresponds to OTM options, negative moneyness corresponds to ITM options and zero moneyness to ATM options.

Depending on whether moneyness or strike is used, the parametric approach models the implied volatility surface through either of the following equations:

**For moneyness:**

$$(2-2) \quad \sigma(MN, \tau) = b_1 + b_2(MN) + b_3(MN)^2 + b_4\tau + b_5(MN)\tau + \varepsilon_{MN,\tau}$$

**For strike:**

$$(2-3) \quad \sigma(E, \tau) = b_1 + b_2(E) + b_3(E)^2 + b_4\tau + b_5(E)\tau + \varepsilon_{E,\tau}$$

In these equations,  $\sigma$  is the implied volatility of an option with strike (E)/moneyness (MN) and time to maturity  $\tau$ .  $b_i$ 's are unknown model coefficients, and  $\varepsilon$ 's are model errors.

The coefficients of this model are estimated by ordinary least-squares (OLS) method and using the observed implied volatilities.  $b_1$  determines the displacement of the parabola with respect to at-the-money options,  $b_3$  determines the wideness of the smile,  $b_4$  determines the maturity effect and  $b_5$  captures the combined effect of maturity and moneyness/strike.

The serious drawback of this model is that if we slice it at different maturities, the resulting slices have always the same shape. On the other hand the linear relationship with time to maturity is just a simplistic way of capturing the maturity effect.

## **2.2 Non-parametric Model**

In the non-parametric model, implied volatilities for unobserved options are interpolated using a nearest neighbour weighted interpolation method.

Assume that we have observed N option prices with strikes and times to maturity of  $(E_i, \tau_i)$ , where  $i=1, \dots, N$ . Then for an unobserved pair of  $(E, \tau)$ , the implied volatility is estimated using the following equation:

$$(2-4) \quad \sigma(E, \tau) = \sum_{k=1}^N \left[ \frac{\sigma_k}{[(\tau_k - \tau)^2 + (E_k - E)^2]^{p/2}} \right] \bigg/ \sum_{k=1}^N \left[ \frac{1}{[(\tau_k - \tau)^2 + (E_k - E)^2]^{p/2}} \right]$$

The parameter p in the above equation determines the relative importance of distant samples. The estimation is generally better when higher values of p are used. For the purpose of this paper, we have used p=8 which was utilized in Borovkova and Permana (2009).

This method works fine if just a few numbers of liquid options are available, but if there are many liquid options in the market, it will have a rather complex shape. Moreover, if we slice the volatility

surface estimated using this approach at different maturities, each slice may have several minima and maxima, and the whole surface may not address the different arbitrage issues properly (More on this in section 3).

## 2.3 Semi-parametric Model

This method combines the attractive features of both parametric and semi-parametric approaches. In this method, a quadratic function is fitted to available option prices at different maturities:

$$(2-5) \quad \sigma_\tau = b_{1,\tau} + b_{2,\tau}(X) + b_{3,\tau}(X)^2 + \varepsilon_\tau$$

First we fix the time to maturity and then apply the above equation to the existing options at that maturity. A semi-parametric method can be fitted either based on strike or moneyness:

**For strike:**

$$(2-6) \quad \hat{\sigma}_i = b_1 + b_2(E_i) + b_3(E_i)^2 + \varepsilon_i$$

**For moneyness:**

$$(2-7) \quad \hat{\sigma}_i = b_1 + b_2(MN_i) + b_3(MN_i)^2 + \varepsilon_i$$

One of the drawbacks of this method is that the number of options at each maturity should be at least three, otherwise we cannot apply equations (2-6) or (2-7). This issue is troublesome especially during the implementation phase, as we have to check whether three options exist at a maturity or not.

After we have built each slice, then we use the bi-cubic spline method to interpolate between different slices and thus estimate the volatility surface.

The parameters of the quadratic functions above can be estimated either by ordinary least-squares method or a weighted least-squares approach. For doing so, first we minimize the sum of squared residuals as in equation (2-8):

$$(2-8) \quad S = \sum_{i=1}^n \omega_i (\hat{\sigma}_i - \sigma_i)^2$$

In case the volatility surface is estimated using moneyness ( $MN$ ) we have:

$$(2-9) \quad MN^T = \begin{bmatrix} 1 & MN_1 & MN_1^2 \\ \vdots & \vdots & \vdots \\ 1 & MN_n & MN_n^2 \end{bmatrix}$$

$$(2-10) \quad \hat{b} = ((MN)W(MN)^T)^{-1}(MN)W\sigma$$

And if the surface is based on strike prices ( $E$ ):

$$(2-11) \quad E^T = \begin{bmatrix} 1 & E_1 & E_1^2 \\ \vdots & \vdots & \vdots \\ 1 & E_n & E_n^2 \end{bmatrix}$$

$$(2-12) \quad \hat{b} = (EWE^T)^{-1}EW\sigma$$

In the above equations, we have

$$(2-13) \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$(2-14) \quad \sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$$

$$(2-15) \quad W = diag\{\omega_1, \dots, \omega_n\}$$

When we use the ordinary least-squares method to estimate the coefficients of equations (2-6) or (2-7), the elements of equation (2-15) become:

$$W = diag\{1, \dots, 1\}$$

And in case we use weighted least-squares approach we may use Liquidity-based or Gaussian methods. Depending on the availability of data, we can use Volume of trades or Open interest as proxies for liquidity of the options. If such data is not available we can give the weights based on how far an option is from being at the money. The calculation and implementation of this approach is further elaborated on in the section 3.



After we have created the volatility slices at each time to maturity, we utilize bi-cubic spline approach to interpolate between different slices:

$$(2-16) \quad p(x, y) = \sum_{j=0}^3 \sum_{i=0}^3 c_{i,j} t^i u^j$$

$$(2-17) \quad p_x(x, y) = \sum_{j=0}^3 \sum_{i=0}^3 i c_{i,j} t^{i-1} u^j$$

$$(2-18) \quad p_y(x, y) = \sum_{j=0}^3 \sum_{i=0}^3 j c_{i,j} t^i u^{j-1}$$

$$(2-19) \quad p_{xy}(x, y) = \sum_{j=0}^3 \sum_{i=0}^3 ij c_{i,j} t^{i-1} u^{j-1}$$

In these equations,  $c_{i,j}$  's are constants and  $u$  and  $t$  are parameters ranging from 0 to 1.  $x$  and  $y$  represent strike/moneyness and time to maturity respectively.

## 3: Methodology

### 3.1 Data collection

As the process of data collection for this sort of research is lengthy and confusing at times, we have provided a detailed description of our efforts so that it would be a contribution to future work. We have used Bloomberg terminal to obtain option prices of crude oil futures traded on New York Mercantile Exchange (NYMEX). In the Bloomberg terminal, function “MFP<GO>” creates access to NYMEX’s Energy Futures list. This list provides real time data regarding the futures contracts traded on this market. To get more specific, we can use “CTM<GO>” to see all the categories of exchange traded contracts in Commodity markets. From this screen, when we choose “Crude Oil” a new screen appears which lists all the exchange traded contracts for crude oil. By setting the proper searching criteria, we will see the list of Crude oil (WTI)’s futures that are currently active on NYMEX and which are suitable for our work.

Table 3-1: Crude oil active future contracts on NYMEX

<b>Ticker</b>	<b>Description</b>	<b>Exchange</b>
CLA Comdty	Crude Oil, WTI	NYMEX Exchange
HBA Comdty	Crude Oil, European	NYMEX Clearport
G9A Comdty	Crude Oil, WTI Asian options	NYMEX Clearport

In Table (3-1), CLA is the active futures contract on light sweet crude oil, HBA is just set up to mirror the CLA contract, but for the sole purpose of pricing European style options and G9A is set up to mirror WTI financial futures to support the average price options.

The next step is to substitute the letter “A” in the above tickers with an appropriate letter in the following table to get the ticker for each month of the year.

Table 3-2: Structure of the tickers used (1<sup>st</sup> part)

	<b>January</b>	<b>February</b>	<b>March</b>	<b>April</b>	<b>May</b>	<b>June</b>	<b>July</b>	<b>August</b>	<b>September</b>	<b>October</b>	<b>November</b>	<b>December</b>
	<b>F</b>	<b>G</b>	<b>H</b>	<b>J</b>	<b>K</b>	<b>M</b>	<b>N</b>	<b>Q</b>	<b>U</b>	<b>V</b>	<b>X</b>	<b>Z</b>
<b>CL</b>	CLF	CLG	CLH	CLJ	CLK	CLM	CLN	CLQ	CLU	CLV	CLX	CLZ
<b>HB</b>	HBF	HBG	HBH	HBJ	HBK	HBM	HBN	HBQ	HBU	HBV	HBX	HBZ
<b>G9</b>	G9F	G9G	G9H	G9J	G9K	G9M	G9N	G9Q	G9U	G9V	G9X	G9Z

In the last step and to create the final ticker for a month of interest, we just have to add the year number based on the table below, followed by “Comdty” function.

Table 3-3: Structure of the tickers used (2nd part)

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023
Suffix	06	07	08	09	10	11	12	13	14*	5	6	7	8	9	0	1	2	3

Below are the tickers of the underlying futures contracts that we have used in this paper:

Table 3-4: Tickers of the underlyings of the 3 datasets

Month	Year	Underlying Futures Contract		
		Dataset 1	Dataset 2	Dataset 3
		<b>CLA</b>	<b>HBA</b>	<b>G9A</b>
November	2014	<b>CLX4</b>	<b>HBX4</b>	<b>G9X4</b>
December	2014	<b>CLZ4</b>	<b>HBZ4</b>	<b>G9Z4</b>
January	2015	<b>CLF5</b>	<b>HBF5</b>	<b>G9F5</b>
February	2015	<b>CLG5</b>	<b>HBG5</b>	<b>G9G5</b>
March	2015	<b>CLH5</b>	<b>HBH5</b>	<b>G9H5</b>
April	2015	<b>CLJ5</b>	<b>HBJ5</b>	<b>G9J5</b>
May	2015	<b>CLK5</b>	<b>HBK5</b>	<b>G9K5</b>
June	2015	<b>CLM5</b>	<b>HBM5</b>	<b>G9M5</b>

One thing to mention here is that the expiry date of the options on above futures is a few days earlier than the maturity of the underlying futures. This issue has been accounted for during the implementation process.

The most complex part of the data collection process was to see what options at which maturities and strikes are linked to the above contracts. Using Bloomberg function “OPT\_CHAIN” in Excel, one can obtain the tickers of all of the options linked to an underlying security. But unfortunately, this function just works real time and we cannot observe what options were linked to an underlying security on a specific date in the past. For doing so we have to build the tickers manually.

If we are looking for call options, we add “C”, and for put options we add “P” to any of the tickers in Table (3-4) followed by the strike price and “Comdty”. As an example, the ticker for an option on a

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\* In 2014, for futures from January to May “14” is used, and for the ones afterwards, “4” is used.

June 2015 futures with a strike of 60\$ becomes CLM5C 60.00 Comdty for call option and CLM5P 60.00 Comdty for put option.

The final step in creating the tickers for the data under consideration is to have a range of strikes to make sure no option contract is ignored in the process. For this purpose, we considered a range from 10 \$ to 200\$, with increments of 0.5 \$. This way we will have a long list of options, but at the same time, we are making sure that everything is included. The advantage of following this procedure is that we somehow replicate the strike prices in the market. As an example, if the starting strike is 40 \$, the next one would be 40.5 \$, then 41 \$ and so on.

So based on our elaborations in the previous paragraphs, we have obtained the following three datasets of options on crude oil futures traded on New York Mercantile Exchange (NYMEX) as of 8<sup>th</sup> October 2014:

*Table 3-5: Datasets*

<b>Dataset</b>	<b>Description</b>	<b>Maturity</b>
No. 1	Prices of American options on crude oil futures	Oct 2014 – May 2015
No. 2	Prices of European options on crude oil futures	Oct 2014 – May 2015
No. 3	Crude oil average price (Asian) option prices	Oct 2014 – May 2015

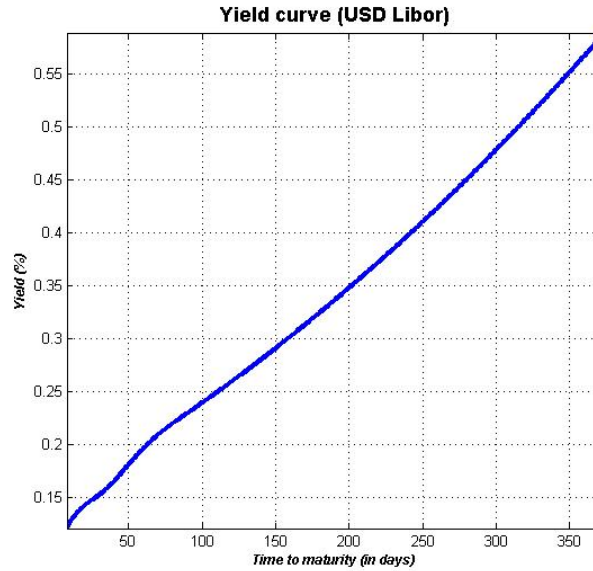
## 3.2 Risk free rate

Risk-free interest rates are required in order to calculate the implied volatilities of our observed options. We have used USD LIBOR rates as proxy for risk-free interest rates to create the yield curve. Cubic spline method is used to fit a smooth curve to the interest rates as of 8<sup>th</sup> October 2014.

*Table 3-6: USD Libor rate as of 8<sup>th</sup> Oct 2014*

<b>Period</b>	<b>Rate</b>
Intraday	0.09000
1 week	0.11730
1 month	0.15180
2 months	0.19600
3 months	0.22910
6 months	0.32415
1 year	0.56730

Figure 3-1: USD Libor yield curve as of 8<sup>th</sup> Oct 2014



### 3.3 Implied volatilities

Implied volatilities for options on futures are calculated by inverting the Black model (1976) and using numerical methods. The Black model is a variant of the famous Black-Scholes option pricing model but its main domains of application include pricing options on futures, bond options, interest rate caps and swaptions. In this paper we have used MATLAB's built-in function "blsimpv" to estimate the implied volatilities for the observed option prices. After implementation of what has been described so far, we are able to apply our models of interest to create the volatility surfaces.

### 3.4 Fitting the parametric model

As stated in the previous section, the parametric implied volatility surface can be created as a function of either strike or moneyness, and the time to maturity. For the purpose of this paper, the results are illustrated based on moneyness.

Implementation of the parametric model is much easier than other approaches since we do not require many data points, and additionally by just estimating a few parameters, we are able to create the whole surface. Another factor, which contributes to the simple application of these models, is the tools that some programming languages like MATLAB provide at our disposal. In MATLAB we can fit a parametric model by using the built-in function "fit" and the library model

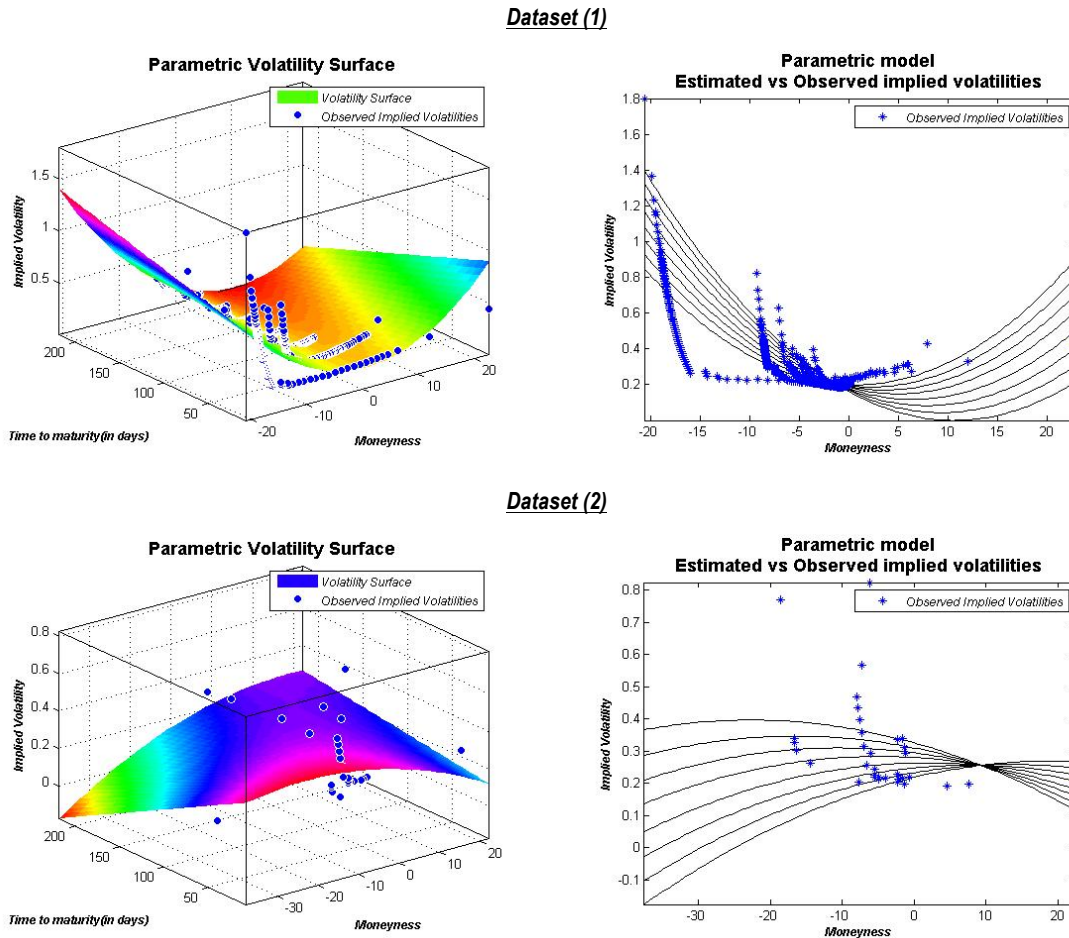
“poly21”. Table (3-7) lists the set of parameters estimated from fitting this model to each of our datasets:

Table 3-7: Parameters of the parametric model

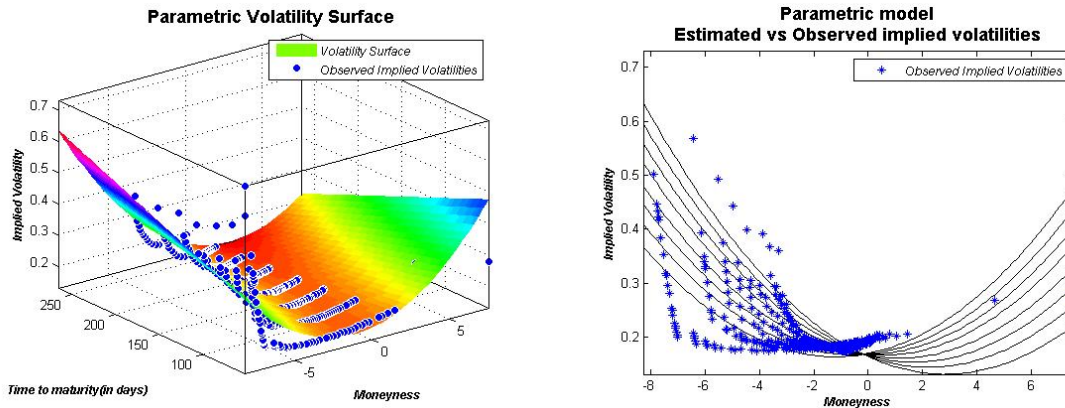
	Dataset 1	Dataset 2	Dataset 3
<b>b1</b>	0.200500	0.330200	0.170300
<b>b2</b>	-0.000747	-0.006709	0.017750
<b>b3</b>	0.001434	-0.000138	0.004082
<b>b4</b>	-0.000183	-0.000513	-0.000026
<b>b5</b>	-0.000135	0.000055	-0.000155

In Figure (3-2), we can see different volatility surfaces and volatility slices that we have fitted to our three datasets under the parametric model:

Figure 3-2: Comparison of volatility surfaces\_Parametric model



### Dataset (3)



As can be observed in Figure (3-2), the parametric model does not seem to fit the observed implied volatilities correctly. There are many outliers and as stated earlier, in this model the volatility slices maintain the same shape across all maturities. The quadratic function of moneyness/strike is assumed to be the same across all maturities and the linear relationship with time to maturity is rather simplistic. As was the case in Borovkova and Permana (2009), the parametric model still seems not to be a suitable model and it is incapable of generating accurate results during a highly volatile market.

### 3.5 Fitting the Non-parametric model

First, we have created a grid of points in two dimensions with the x-axis being the strikes and y-axis being the times to maturity. The strikes start from the minimum strike price of the dataset used and continues to the maximum strike. The steps are set to 0.5. This way we can somehow follow the pattern of the existing strike prices in the market.

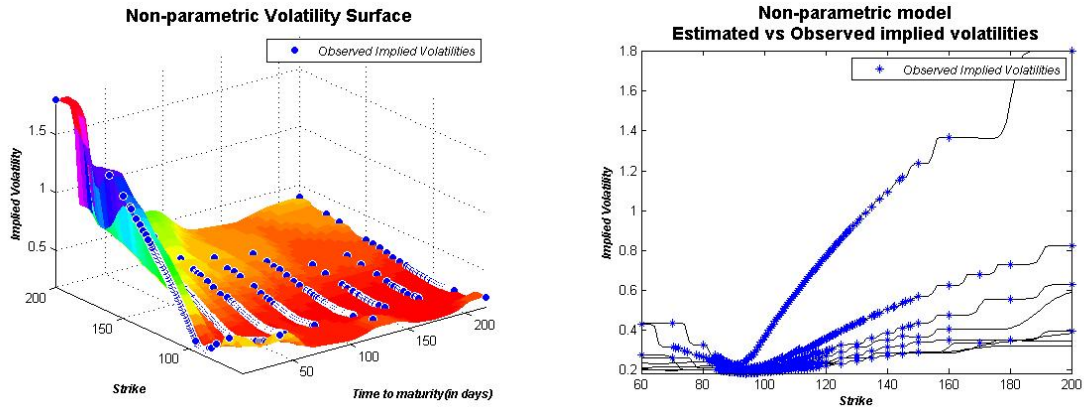
The y-axis has the same number of steps as the x-axis, which can be altered. Then we have applied equation (2-4) to each point in the grid, and using the strike, time to maturity and implied volatility vectors of the observed option prices. This way, we have a three-dimension array in which each element is the implied volatility corresponding to the respective strike and time to maturity.

Just note that the non-parametric equation should be applied to data points that are not observed in the market. That is, if the strike and time to maturity of a grid point coincides with an observed point, instead of applying equation (2-4) to compute the respective implied volatility; we should just substitute the observed implied volatility. The reason is that the denominators in the equation become zero and consequently we would get errors.

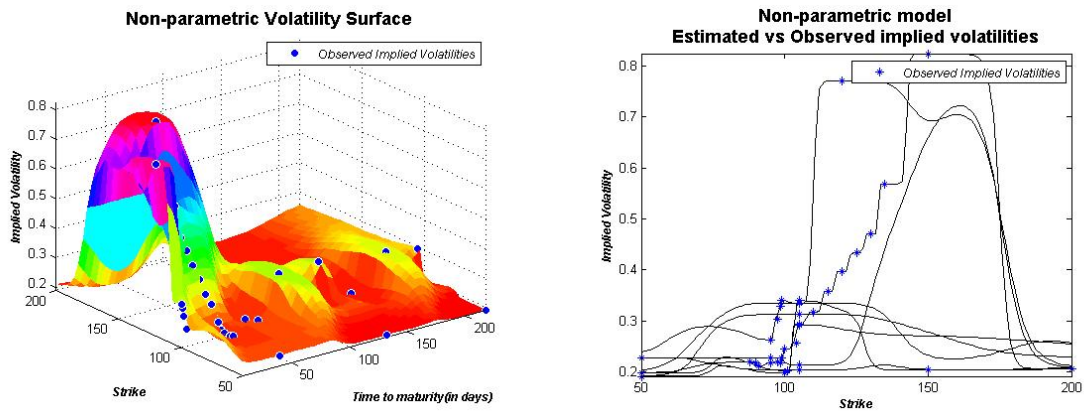
The plots in Figure (3-3) illustrate how exactly the volatility surfaces under this method fit the observed implied volatilities, but at the expense of having very complex shapes.

Figure 3-3: Comparison of volatility surfaces\_Non-parametric model

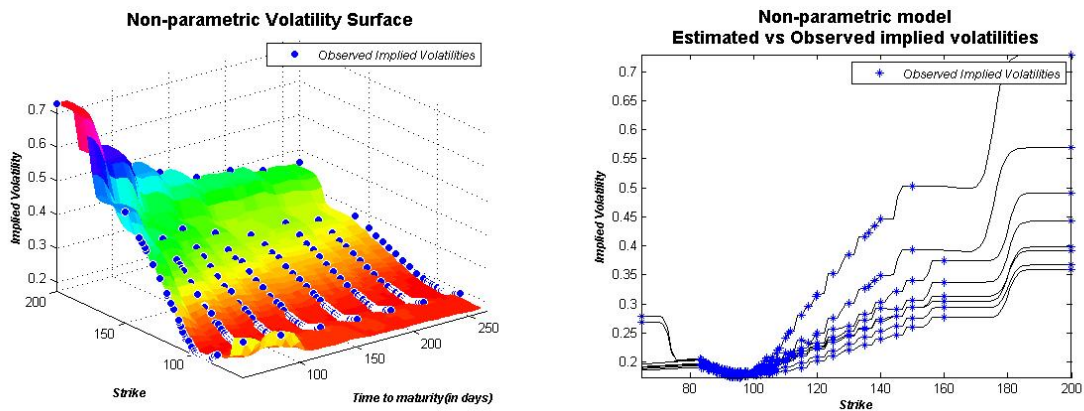
**Dataset (1)**



**Dataset (2)**



**Dataset (3)**





Our results from application of the non-parametric model in a highly volatile market are consistent with earlier findings in Borovkova and Permana (2009). The non-parametric approach lacks generalization power as it just attempts to fit the observed implied volatilities exactly. Consequently and as can be noted in Figure (3-3), we may have several minima and/or maxima in cross-sections of the surface along the maturity axis. We call each of these cross-sections a volatility slice. As stated in Borovkova and Permana (2009), the multimodality of the volatility slices translates into multimodality of option prices versus strike. Multimodal asset price distributions cannot occur under the risk-neutral probability measure. The multimodality of the volatility slices is inconsistent with risk aversion in the market, hence destroys the correct behaviour of the option's Vega versus strike.

### 3.6 Fitting the Semi-parametric model

In this paper the semi-parametric volatility surface is presented as a function of time to maturity and strike price. In this method, we initially need to categorize our data points based on their maturities. Then we can fit equation (2-6) to each of these categories in order to get the parameters of the volatility slices. This way we have fitted a quadratic function of different strikes at each maturity. As described in section 2, there are two methods to estimate the quadratic functions' parameters: ordinary least-squares and weighted least-squares.

Under the ordinary least-squares approach, all the elements of the equation (2-15) become "1". But if we take the weighted least-squares approach, we would have two methods to deal with the situation: Liquidity-based weights or Gaussian.

Liquidity-based weights are derived using either daily open interests or volume of trades. Under this method, for each category of options with the same maturity, each data point is weighted based on its liquidity factor relative to the whole group. Obviously, at-the-money options that are more liquid will get a higher weight than out or in-the-money contracts. As a result the quadratic function would become a better fit for at-the-money options.

In case we do not have access to the open interest as well as the volume data, we can utilize the Gaussian method. Under this method, each option is weighted based on how far it is from being at-the-money. To calculate these weights first we determine at-the-money strike ( $E^*$ ) of the options in each category. Since we are studying the futures, at-the-money strike in each category equals to the futures price in that category. Then we compute the distance of each option's strike from at-the-money strike price ( $d_i = E_i - E^*$ ). Finally, the weight of each option will equal to  $\varphi_s(d_i)$ , in which ( $s$ ) is the empirical standard deviation of  $d_i$  's. The sets of quadratic function parameters for our datasets are displayed in the following tables:

Table 3-8: Parameters of the semi-parametric model (Dataset 1)

CLA		Maturity							
		1	2	3	4	5	6	7	8
OLS	<b>b3</b>	-1.212766	0.452226	0.189898	0.477041	0.565004	0.427259	0.492701	0.331865
	<b>b2</b>	0.016929	-0.006367	-0.001641	-0.006630	-0.007839	-0.005582	-0.006716	-0.003172
	<b>b1</b>	-0.000005	0.000046	0.000020	0.000040	0.000042	0.000033	0.000038	0.000018
Gaussian	<b>b3</b>	2.375934	1.741131	1.307842	1.150852	1.000867	0.709714	0.692364	0.758903
	<b>b2</b>	-0.053397	-0.032404	-0.023204	-0.019799	-0.016591	-0.011102	-0.010628	-0.011777
	<b>b1</b>	0.000333	0.000173	0.000123	0.000103	0.000086	0.000060	0.000057	0.000061
Liquidity	<b>b3</b>	-1.783459	-0.077874	0.524895	0.613289	0.795013	0.706657	0.680142	0.493844
	<b>b2</b>	0.024190	0.001668	-0.008061	-0.009207	-0.012625	-0.010434	-0.010170	-0.006555
	<b>b1</b>	-0.000026	0.000016	0.000050	0.000052	0.000067	0.000053	0.000053	0.000035

Table 3-9: Parameters of the semi-parametric model (Dataset 2)

HBA		Maturity			
		1	2	3	4
OLS	<b>b3</b>	-0.824725	-0.483553	-0.019506	1.890681
	<b>b2</b>	0.004347	0.011581	0.005691	-0.051054
	<b>b1</b>	0.000075	-0.000036	-0.000028	0.000341
Gaussian	<b>b3</b>	-0.812376	0.737477	0.025356	1.890681
	<b>b2</b>	0.004116	-0.015125	0.004303	-0.051054
	<b>b1</b>	0.000076	0.000102	-0.000019	0.000341
Liquidity	<b>b3</b>	-0.619237	-0.078948	0.188888	1.890681
	<b>b2</b>	0.000555	0.001873	0.000214	-0.051054
	<b>b1</b>	0.000092	0.000020	-0.000001	0.000341

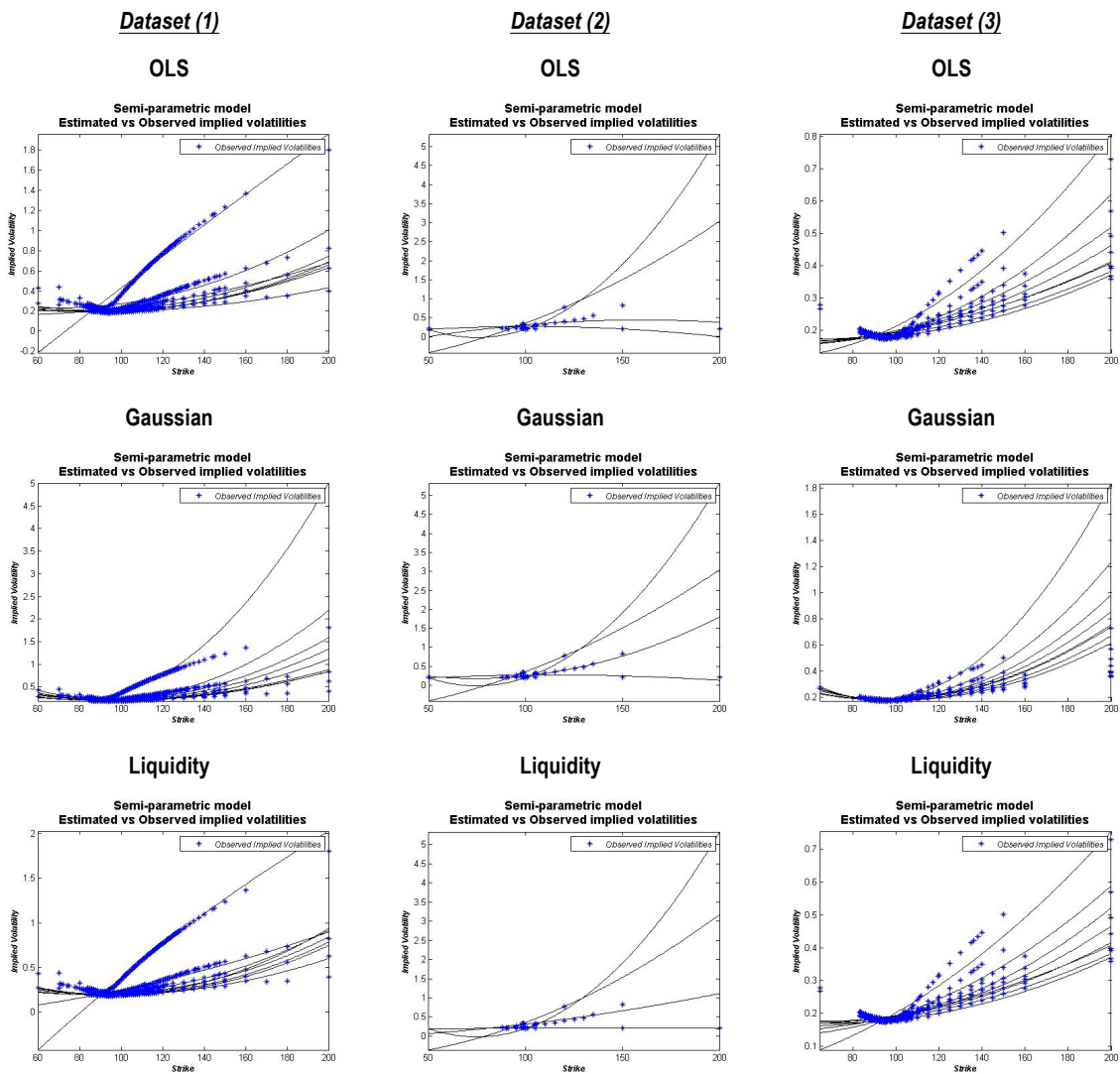
Table 3-10: Parameters of the semi-parametric model (Dataset 3)

G9A		Maturity							
		1	2	3	4	5	6	7	8
OLS	<b>b3</b>	0.154587	0.298086	0.202897	0.207500	0.165080	0.154112	0.193613	0.234800
	<b>b2</b>	-0.002104	-0.003596	-0.001720	-0.001512	-0.000675	-0.000494	-0.001059	-0.001716
	<b>b1</b>	0.000027	0.000026	0.000016	0.000014	0.000010	0.000009	0.000010	0.000012
Gaussian	<b>b3</b>	1.368725	1.041886	0.885555	0.773626	0.650082	0.642999	0.620838	0.597012
	<b>b2</b>	-0.025896	-0.018100	-0.014500	-0.012150	-0.009827	-0.009667	-0.009051	-0.008501
	<b>b1</b>	0.000141	0.000095	0.000075	0.000063	0.000052	0.000051	0.000046	0.000043

G9A		Maturity							
		1	2	3	4	5	6	7	8
Liquidity	<b>b3</b>	-0.005560	0.161870	0.269166	0.254614	0.207084	0.137569	0.180342	0.225908
	<b>b2</b>	0.000353	-0.001493	-0.002771	-0.002266	-0.001352	-0.000283	-0.000896	-0.001613
	<b>b1</b>	0.000017	0.000018	0.000020	0.000017	0.000012	0.000008	0.000010	0.000012

The semi-parametric implied volatility functions (volatility slices) are illustrated in Fig (3-4). Note how the implied volatility functions have distinctly different shapes at different maturities.

Figure 3-4: Comparison of volatility slices\_Semi-parametric model



Based on the plots in Figure (3-4), we can see that for:

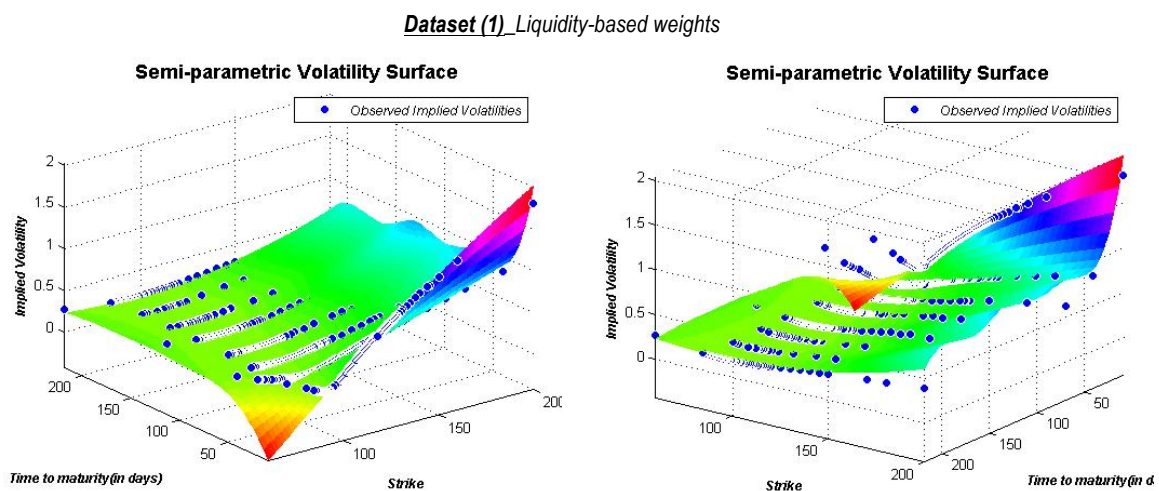
- Dataset (1), the liquidity-based and OLS approaches provide better fits than the Gaussian approach. Options in Dataset (1) are more liquid than the ones in other datasets.
- Dataset (2), the Gaussian approach is the best fit. Options in this category are rarely traded, thus their prices are less reliable.
- Dataset (3), again the Gaussian approach provides a better fit. But in this case, outliers are more noticeable than the other two datasets.
- The one thing that is common in all the cases above is that the semi-parametric model provides a better fit to at-the-money options. But as we move away from the at-the-money strike, we see that this model does not calibrate to the given implied volatilities well.

After we have fitted the quadratic functions to different classes of options and have built the volatility slices at different maturities, we interpolate between these slices using the bi-cubic spline approach to create the volatility surface.

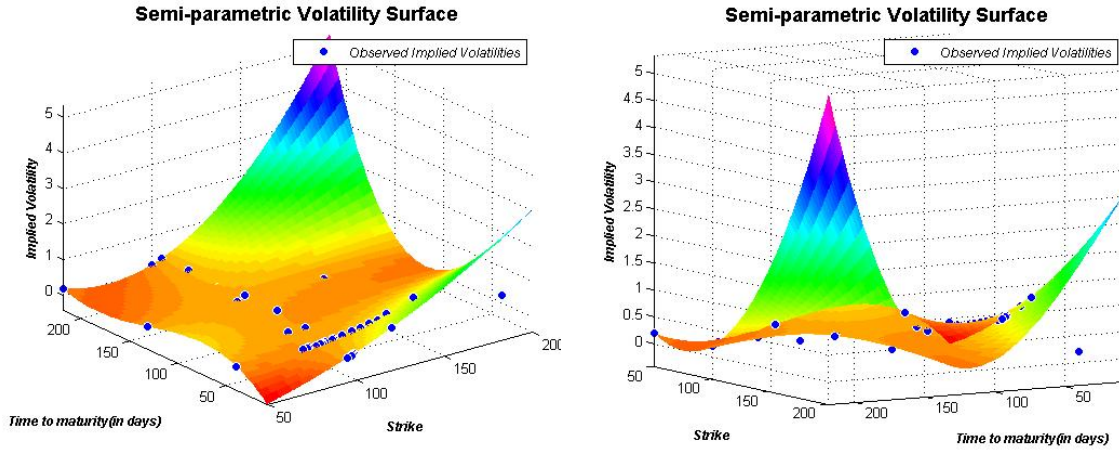
The most important part in fitting the semi-parametric model is when the volatility slices are calibrated at each maturity. It is in this stage that we can figure out whether our model is a good fit for our data or not. The stage of bi-cubic spline interpolation does not make any difference to the slices that are already created. This stage just connects the slices to build a smooth surface.

In Figure (3-5) we can observe different instances of application of the semi-parametric model to our three datasets. Note that the plots on the right hand side are the same as the ones on the left but at a different angle.

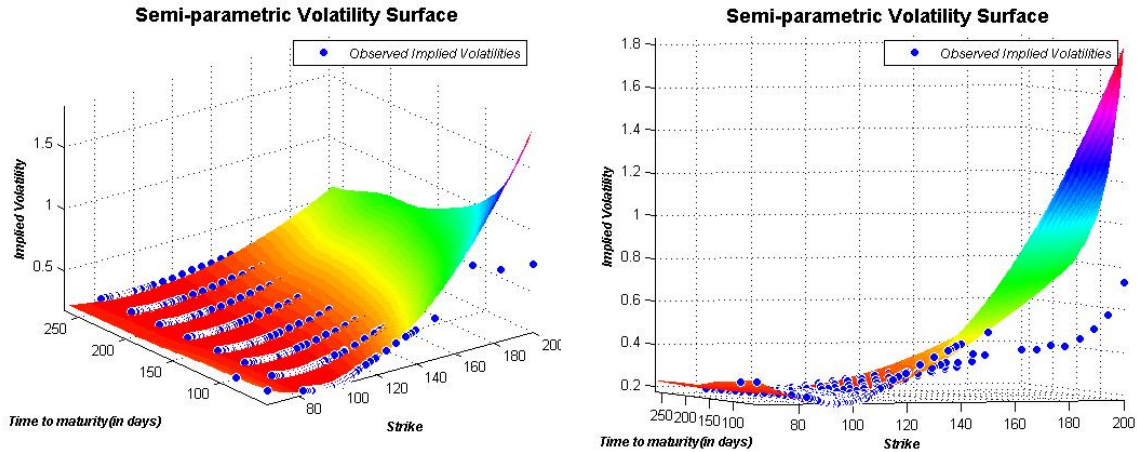
Figure 3-5: Comparison of volatility surfaces\_Semi-parametric model



### Dataset (2)\_Gaussian weights



### Dataset (3)\_Gaussian weights



Our visual observation suggests that the semi-parametric model does not perform well in a highly volatile market. In reality we cannot force the market to behave in a specific manner, but the assumptions of this model restrict the volatility slices to certain shapes. One important point that needs to be mentioned here is that in our datasets we have excluded the options that exist in the market but not traded. We believe prices of these options, hence their implied volatilities are not reliable. If we were to include those options in our applications, we would have faced chaos in our volatility surfaces; too many outliers and unrealistic shape of the surfaces. The other drawback of this model is that the arbitrage-free conditions are not set. Even if the model provided a good fit, how could we determine whether the resulting surface was free of arbitrage or not?

## 3.7 Further analysis

In order to confirm our observations so far, we estimate the mean and standard deviation of our models' residuals and compare them with the metrics that were previously presented in

Borovkova and Permana (2009). Table (3-11) is taken from Borovkova and Permana (2009) and in Table (3-12), we have listed the results of our estimations.

Table 3-11: Mean and standard deviation of residuals, from Borovkova and Permana (2009)

Method	Dataset 1	Dataset 2	Dataset 3
<b>Parametric</b>	0.0096	0.0036	0.0090
	(0.0067)	(0.0034)	(0.0068)
<b>Semi-parametric (ordinary l.s.)</b>	0.0038	0.0016	0.0013
	(0.0061)	(0.0019)	(0.0037)
<b>Semi-parametric (Gaussian weighted l.s.)</b>	0.0044	0.0024	0.0011
	(0.0072)	(0.0070)	(0.0075)
<b>Semi-parametric (liquidity-based weighted l.s.)</b>	0.0048	0.0018	0.0011
	(0.0065)	(0.0021)	(0.0064)

In the table above, the Dataset (1) comprises European options on Crude oil futures and Dataset (2) contains the Average price options. These metrics are estimated based on the application of the three models that we have reviewed in this paper, but the timing of the datasets are different. As stated in Borovkova and Permana (2009), the figures in parentheses are the standard deviations of residuals and the other figures are the mean of residuals. For comparison purposes, the Dataset (1) in the above table corresponds to the Dataset (2) in our paper and the Dataset (2) corresponds to Dataset (3) in this project.

Table 3-12: Mean and standard deviation of residuals of three datasets

Method	Dataset (1)		Dataset (2)		Dataset (3)	
	Mean	STD	Mean	STD	Mean	STD
<b>Parametric</b>	1.65E-16	0.107001	-1.54E-18	0.136917	5.96E-17	0.041923
<b>Semi-parametric (OLS)</b>	3.24E-17	0.097058	-1.71E-15	0.121130	4.91E-17	0.036738
<b>Semi-parametric (Gaussian)</b>	0.017831	0.114362	-0.091162	0.515141	0.008098	0.045297
<b>Semi-parametric (Liquidity-based)</b>	-0.004638	0.244390	-0.067187	0.139537	-0.000330	0.041050

Comparison of Table (3-11) with Table (3-12) suggests that the standard deviations of the residuals have significantly increased when our data is taken from a highly volatile market. Furthermore both our visual observations and the metrics raise serious questions regarding applicability of such local volatility models to option data on crude oil futures.

Figure 3-6: Daily log-return of Crude oil futures

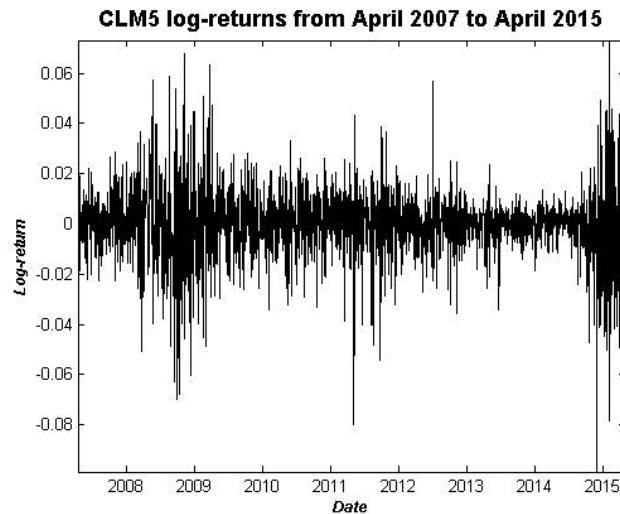
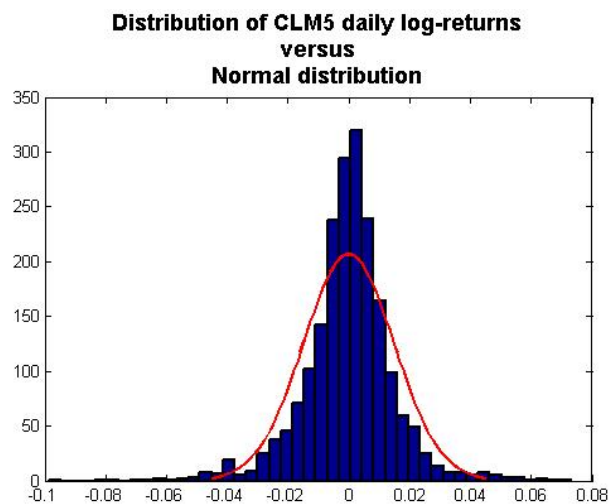


Figure (3-6) displays the daily log-returns of the June 2015 maturity crude oil futures since the first trade in April 2007. As we can see, large moves follow large moves and small moves follow small moves. In other words, we observe volatility clustering in this data.

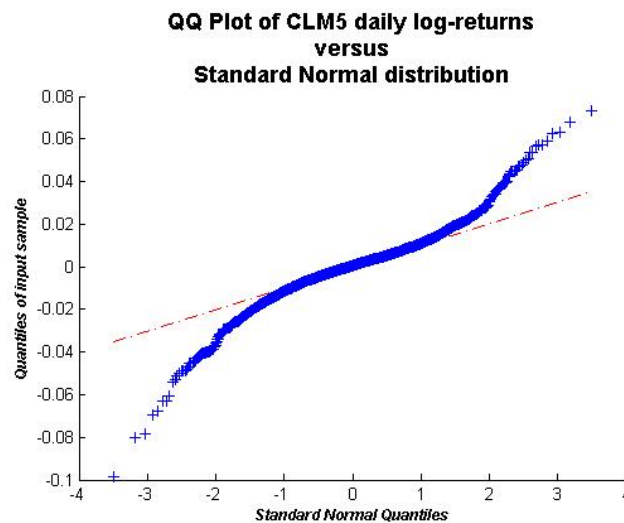
In Figure (3-7), we have plotted the distribution of the above data against normal density function. We observe that the empirical distribution has higher peak and fatter tails compared to the normal distribution.

Figure 3-7: Empirical distribution of futures' daily log-returns vs normal distribution



In Figure (3-8), the Q-Q plot compares the quantiles of the above dataset to the quantiles of the standard normal distribution. In this plot we can observe how the tails of the distribution of log-returns are extremely different from the standard normal distribution. The plot would have been a straight line if the data followed a normal distribution.

*Figure 3-8: QQ Plot of futures' daily log-returns vs standard normal distribution*



Fat tails and the high central peak are characteristics of mixtures of distributions with different variances. The volatility clustering feature implies that volatility (or variance) is auto-correlated (Gatheral, 2006). Our previous findings and the results presented here motivate us to consider volatility as a random variable. Consequently it would be valuable to consider whether stochastic volatility models such as the SVI parameterization of volatility surfaces that was first introduced at Merrill Lynch in 1999 would be a better fit for the options on crude oil futures?



## **4: Conclusion and future work**

In this paper, we tested three local volatility models using three sets of option prices on crude oil futures during a high volatility period. We have demonstrated that such models are no longer capable of adequately fitting the observed data in the crude oil market. We have performed our tests using crude oil option data in 2014 and beginning of 2015. Based on our investigation, we have concluded that a parametric, a non-parametric and a semi-parametric method of implied volatilities do not give valid results when we are dealing with high volatility in the underlying futures.

Moreover, we have shown that the daily log-returns of crude oil futures display volatility clustering and their empirical distribution displays higher peak and fatter tails compared to the normal density function which leads to considering the volatility itself as a stochastic process. This fact raises the question of whether local volatility models in general are still appropriate for crude oil options or should we move towards using stochastic volatility models. We leave further investigation of this issue for future research.

## **Appendices**

## Appendix A: MATLAB Codes

Table 4-1: MATLAB\_Yield curve using LIBOR rates

```
%% Term Structure:
CurrentDate='08-Oct-2014';
Data=[0.09000,0.11730,0.15180,0.19600,0.22910,0.32415,0.56730]/100;
Dates = daysadd(CurrentDate,...
    [0 7 (1/12)*360 (2/12)*360 (1/4)*360 (1/2)*360 360],2);
irdc=IRDataCurve('Zero',CurrentDate,Dates,Data,'InterpMethod','spline');

%-----
% Zero Rate Vector:
ZeroRates=irdc.getZeroRates(Exp_Date);
```

Table A-2: MATLAB\_Estimation of implied volatilities

```
%% Implied Volatility:
IV=blkimpv(FutPrice, Strike, ZeroRates, Tenor_y,OptPrice,[],[],{'call'});
```

Table A-3: MATLAB\_Calculation of moneyness

```
%% Moneyness:
Moneyness=log(FutPrice./Strike)./(IV.*Tenor_y);
```

Table A-4: MATLAB\_Calculation of weights for the semi-parametric model

```
function W=Weightings(OpenInt,Vol,Tenor,Strike,FutPrice,method,InputType)
% This function calculates the weights to be used in the Semi-parametric
% model to estimate the coefficients of the quadratic functions.

% Inputs:
% OpenInt: Vector of Open interests of options on a specific date.
% Vol: Vector of trading volume of options on a specific date.
% Tenor: Vector of the time to maturities of the option data corresponding
% to the Strike vector.
% K: Vector of either strike prices or moneynesses.
% FutPrice: Vector of underlying futures prices.
% method: A string indicating the method to be used to estimate the
% quadratic functions' coefficients. It can be any of the followings:
% 'Liquidity_OpenInterest', 'Liquidity_Volume', 'OLS' and 'Gaussian'.
% InputType: A string, either 'Strike' or 'Moneyness'. It determines
```

```

% whether the elements of vector 'K' are Strike prices or Moneyness of
% options.
%
% Outputs:
% A cell containing square matrices. Each square matrix correspond to a
% specific maturity. These matrices are diagonal with their elements being
% the weights to be used in the Semi-parametric model.

%-----
% Determine the unique maturities and number of them:
TTM=unique(Tenor);NofMat=length(TTM);

%-----
% Preallocate the required cell:
W=cell(1,1,length(TTM));

%-----
% Calculate the weights based on each method:
switch method
    case 'Liquidity_OpenInterest'
        for i=1:NofMat;
            W{i}=diag(OpenInt((Tenor==TTM(i)))/sum(OpenInt((Tenor==TTM(i)))));
        end

    case 'Liquidity_Volume'
        for i=1:NofMat;
            W{i}=diag(Vol((Tenor==TTM(i)))/sum(Vol((Tenor==TTM(i)))));
        end

    case 'OLS'
        for i=1:NofMat;
            W{i}=eye(sum(Tenor==TTM(i)));
        end

    case 'Gaussian'
        switch InputType
            case 'Strike'
                d=(Strike-FutPrice);
            case 'Moneyness'
                d=(Strike);
        end
        s=std(d);
        NormWeights=normpdf(d,0,s);

        for i=1:NofMat;
            W{i}=diag(NormWeights(Tenor==TTM(i)));
        end
end
end

```

Table 4-5: MATLAB\_Parametric model

```
function [IVParam_MN_me,IVParam_MN_stde,IVParam_K_me,...
    IVParam_K_stde]=IVParametric(Strike,Moneyness,IV,Tenor)
% This function fits a volatility surface to option data using the
% Parametric model.

% Inputs:
% Strike: Vector of strike prices of options.
% Moneyness: Vector of moneyness of each option.
% IV: Vector of implied volatilities corresponding to Strikes and Tenors.
% Tenor: Vector of the time to maturities of the option data corresponding
% to the Strike vector.
%
% Outputs:
% Two plots of the volatility surfaces fitted to the input data.
% Mean and Standard deviation of residuals.

%-----
% Fit the Parametric model based on either Moneyness or Strike prices:
[Parametric_MN,~,output_MN]=fit([Moneyness,Tenor],IV,'poly21');
[Parametric_K,~,output_K]=fit([Strike,Tenor],IV,'poly21');

%-----
% Plot the volatility surface and the observed implied volatilities based
% on Moneyness:
figure
plot(Parametric_MN, [Moneyness,Tenor], IV);
colormap(hsv);
shading flat
title('Parametric Volatility Surface','FontWeight','bold','FontSize',14);
xlabel('Moneyness','FontWeight','bold','FontSize',9,'FontAngle','italic');
ylabel('Time to maturity(in days)','FontWeight','bold','FontSize',9,...
    'FontAngle','italic');
zlabel('Implied Volatility','FontWeight','bold','FontSize',9,...
    'FontAngle','italic');
legend({'Volatility Surface';'Observed Implied Volatilities'},...
    'Location','Northeast','FontAngle','italic','FontSize',9);
axis tight

%-----
% Plot the volatility surface and the observed implied volatilities based
% on Strike prices:
figure
plot(Parametric_K, [Strike,Tenor], IV);
colormap(hsv);
shading flat
title('Parametric Volatility Surface','FontWeight','bold','FontSize',14);
xlabel('Strike','FontWeight','bold','FontSize',9,'FontAngle','italic');
```

```

ylabel('Time to maturity(in days)','FontWeight','bold','FontSize',9,...
      'FontAngle','italic');
xlabel('Implied Volatility','FontWeight','bold','FontSize',9,...
      'FontAngle','italic');
legend({'Volatility Surface';'Observed Implied Volatilities'},...
      'Location','Northeast','FontAngle','italic','FontSize',9);
axis tight

%-----
% Calculate the mean and standard deviation of residuals:
IVParam_MN_me=mean(output_MN.residuals);
IVParam_MN_stde=std(output_MN.residuals);
IVParam_K_me=mean(output_K.residuals);
IVParam_K_stde=std(output_K.residuals);

end

```

*Table 4-6: MATLAB\_Non-parametric model*

```

function IVNonParametric(Strike,TM,IV,p,MaturityType)
% This function fits a volatility surface to option data
% using the Non-parametric model (nearest-neighbor weighted interpolation
% method.)

% Inputs:
% Strike: Vector of strike prices of option data
% Tenor: Vector of the time to maturities of the option data corresponding
% to the Strike vector.
% IV: Vector of implied volatilities corresponding to Strikes and Tenors.
% P: P is just a quantity. It determines the relative importance of distant
% samples. The approximation is better with higher values of "p".
% MaturityType: A string, either 'd' or 'y'. It determines whether the
% input Tenor vector is in years or in days.
%
% Output:
% A plot of the volatility surface fitted to the input data.

%-----
% Set the convention to be used(360 or 365 days):
year=365;

%-----
% This part determines the increments required for the calculations below:
switch MaturityType
    case 'd'
        increment=1;
    case 'y'

```

```

        increment=1/year;
end

%-----
% Create the required grid for interpolation in the following sections:
Tenor=min(TTM):increment:max(TTM);
K=linspace(min(Strike),max(Strike),length(Tenor));
[X,Y] = meshgrid(Tenor,K);
ImpliedVol=nan(size(X));
[idx,jdx]=size(X);

%-----
% Interpolate and estimate the implied volatilities at each grid point:
for i=1:idx;
    for j=1:jdx;
        if sum((TTM==X(i,j)).*(Strike==Y(i,j)))==0;
            d=((TTM-X(i,j)).^2+(Strike-Y(i,j)).^2).^(p/2);
            ImpliedVol(i,j)=sum(IV./d)/sum(1./d);
        else
            ImpliedVol(i,j)=sum(IV.*((TTM==X(i,j)).*(Strike==Y(i,j))));
        end
    end
end

%-----
% Plot the volatility surface along with the observed implied volatilities:
figure
surf(X,Y,ImpliedVol);
colormap(hsv);
shading flat
hold on
h=plot3(TTM,Strike,IV,'o','MarkerEdgeColor','w','MarkerFaceColor','b',...
    'MarkerSize',7);
title('Non-parametric Volatility Surface','FontWeight','bold',...
    'FontSize',14);
xlabel('Time to maturity(in days)','FontWeight','bold','FontSize',9,...
    'FontAngle','italic');
ylabel('Strike','FontWeight','bold','FontSize',9,'FontAngle','italic');
zlabel('Implied Volatility','FontWeight','bold','FontSize',9,...
    'FontAngle','italic');
hleg=legend(h,'Observed Implied Volatilities');
set(hleg,'Location','Northeast','FontAngle','italic','FontSize',9);
axis tight
hold off

end

```

Table 4-7: MATLAB\_Semi-parametric model

```
function [method,IVSmParam_K_me,IVSmParam_K_stde]=IVSemiParametric_Strike(...
    OpenInt,Vol,Tenor,Strike,FutPrice,IV,method,MaturityType)
% This function fits a volatility surface to option data using the
% Semi-parametric model. In this model, a quadratic function is fitted to
% implied volatilities at each maturity to create the volatility slices.
% Then these slices are connected to each other by bi-cubic spline
% interpolation.

% Inputs:
% OpenInt: Vector of Open interests of options on a specific date.
% Vol: Vector of trading volume of options on a specific date.
% Tenor: Vector of the time to maturities of the option data corresponding
% to the Strike vector.
% Strike: Vector of strike prices of options.
% FutPrice: Vector of underlying futures prices.
% IV: Vector of implied volatilities corresponding to Strikes and Tenors.
% method: A string indicating the method to be used to estimate the
% quadratic functions' coefficients. It can be any of the followings:
% 'Liquidity_OpenInterest', 'Liquidity_Volume', 'OLS' and 'Gaussian'.
% MaturityType: A string, either 'd' or 'y'. It determines whether the
% input Tenor vector is in years or in days.
%
% Outputs:
% A plot of the volatility surface fitted to the input data.
% Mean and Standard deviation of residuals.

%-----
% Determine the unique maturities and number of them:
TTM=unique(Tenor);NofMat=length(TTM);

%-----
% Preallocate the required cells and arrays:
E=cell(1,1,NofMat);Sig=cell(1,1,NofMat);InterpIV=cell(1,1,NofMat);
Residuals=nan(length(Tenor),1);b=nan(3,NofMat);

%-----
% Set the step sizes for the grid of data in the following sections:
switch MaturityType
    case 'd'
        increment=5;
    case 'y'
        increment=5/year;
end

%-----
% Weights:
W=Weightings(OpenInt,Vol,Tenor,Strike,FutPrice,method,'Strike');
```



```

%-----
% Estimate the coefficients of the quadratic functions(for each volatility
% slice):
for i=1:NofMat;
    SelectedOptions=(Tenor==TTM(i));
    CountOptions=sum(SelectedOptions);
    if CountOptions<3;
        continue
    end
    E{i}=nan(3,CountOptions);
    E{i}(1,:)=1;
    E{i}(2,:)=Strike(SelectedOptions);
    E{i}(3,:)=E{i}(2,:).^2;
    Sig{i}=nan(CountOptions,1);
    Sig{i}=IV(SelectedOptions);
    b(:,i)=E{i}*W{i}*E{i}'\E{i}*W{i}*Sig{i};
    % Residuals:
    InterpIV{i}=b(:,i)'*[E{i}(1,:);E{i}(2,:);E{i}(3,:)];
    Residuals(SelectedOptions)=Sig{i}-InterpIV{i}';
end

%-----
% Remove the NaNs from residuals and estimate the mean and standard
% deviation of residuals:
Residuals=Residuals(~isnan(Residuals));
IVSmParam_K_me=mean(Residuals);
IVSmParam_K_stde=std(Residuals);

%-----
% Remove the NaNs from the results. NaNs appear when there are less than 3
% options at each maturity:
TTM=TTM(~isnan(b(1,:))');
b=b(:,~isnan(b(1,:)));

%-----
% Build the volatility slices with the coefficients estimated in the
% previous sections:
K=min(Strike):0.5:max(Strike);
X=[ones(1,length(K));K;K.^2];

SigSlice=b'*X;

%-----
% Plot the volatility slices and the observed implied volatilities:
figure('Name',method)

plot(K,SigSlice,'k','LineWidth',1.2);
hold on

```

```

h=plot(Strike,IV,'*');
xlabel('Strike','FontWeight','bold','FontSize',9,'FontAngle','italic');
ylabel('Implied Volatility','FontWeight','bold','FontSize',9,...
    'FontAngle','italic');
title({'Semi-parametric model';...
    'Estimated vs Observed implied volatilities'},'FontWeight',...
    'bold','FontSize',14);
hleg=legend(h,'Observed Implied Volatilities');
set(hleg,'Location','Northeast','FontAngle','italic','FontSize',9);
axis tight
hold off

%-----
% Creat the grid required in order to interpolate between volatility
% slices:
[Xa,Ya]=meshgrid(K,TTM);
[Xq,Yq]=meshgrid(K,min(TTM):increment:max(TTM));
Vfine=interp2(Xa,Ya,SigSlice,Xq,Yq,'spline');

%-----
% Plot the volatility surface and the observed implied volatilities:
figure('Name',method)

surf(Xq,Yq,Vfine);
colormap(hsv);
shading flat
hold on
h=plot3(Strike,Tenor,IV,'o','MarkerEdgeColor','w','MarkerFaceColor',...
    'b','MarkerSize',7);
title('Semi-parametric Volatility Surface','FontWeight','bold',...
    'FontSize',14);
xlabel('Strike','FontWeight','bold','FontSize',9,'FontAngle','italic');
ylabel('Time to maturity(in days)','FontWeight','bold','FontSize',9,...
    'FontAngle','italic');
zlabel('Implied Volatility','FontWeight','bold','FontSize',9,...
    'FontAngle','italic');
hleg=legend(h,'Observed Implied Volatilities');
set(hleg,'Location','Northeast','FontAngle','italic','FontSize',9);
axis tight
grid on
hold off

end

```

Table A-8: MATLAB\_Semi-parametric model (2<sup>nd</sup> approach)

```
% Preallocate the required cell:
fitobject1=cell(1,1,NofMat);

%-----
% Fit the quadratic function at each maturity:
for i=1:NofMat;
    SelectedOptions=(Tenor_y==TTM(i));
    if sum(SelectedOptions)<3;
        TTM(i)=nan;
        continue
    end
    fitobject1{i}=fit(Strike(SelectedOptions),IV(SelectedOptions),...
        'poly2','Weights',diag(W{i}));
end

%-----
% Remove non-empty elements from the results of the above loop:
TTM=TTM(~isnan(TTM));
fitobject1=fitobject1(~cellfun(@isempty,fitobject1));

%-----
% Build the volatility slices:
K=min(Strike):0.5:max(Strike);
NofMat=length(TTM);
IVSslice=nan(NofMat,length(K));

for i=1:NofMat;
    IVSslice(i,:)=feval(fitobject1{i},K);
end

%-----
% Prepare the data points from volatility slices to create the volatility
% surface:
[K,TTM,IVSslice]=prepareSurfaceData(K,TTM,IVSslice);

% Fit the volatility surface to the slices using the bicubic spline
% interpolation method:
[fitobject2,gof2,output2]=fit([K,TTM],IVSslice,'cubicinterp');
```

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